Journal of Sound and Vibration (1999) **220**(3), 451–468
Article No. jsvi.1998.1958, available online at http://www.idealibrary.com on IDEAL

A NEW APPROACH FOR DETERMINING THE NATURAL FREQUENCIES AND MODE SHAPES OF A UNIFORM BEAM CARRYING ANY NUMBER OF SPRUNG MASSES

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(*Received* 28 *April* 1998, *and in final form* 26 *August* 1998)

In theory, one may obtain five equations from each attaching point of a spring–mass system and two boundary–equations from each end of the uniform beam. Hence, for a uniform beam carrying n spring–mass systems, simultaneous equations of the form $[\bar{B}]\{\bar{C}\} = \mathbf{0}$ will be obtained. The solutions of $|\bar{B}| = \mathbf{0}$ (where $|\cdot|$ represents a determinant) give the natural frequencies of the "constrained" beam and the substitution of each corresponding values of $\bar{C}_j$ ($j = 1 \sim 4$) into the eigenfunction will define the associated mode shapes. While the foregoing theory is simple, the lengthy explicit mathematical expressions become impractical if the total number of spring–mass systems is larger than "two". For this reason, it was applied to do the free vibration analysis of a uniform beam carrying "one" spring–mass system only in the existing literature. The purpose of this paper is to present a numerical technique to apply the foregoing theory to obtain the exact solutions for the lowest several natural frequencies and mode shapes of a uniform beam carrying "any number of" spring–mass systems with various boundary conditions. To this end, each integration constant C_{vi} and each mode displacement Z_v ($v = 1 \sim n$, $i = 1 \sim 4$) at the attaching point and the two ends of the beam are considered as nodal displacements of a finite beam element and are assigned an appropriate degree of freedom (dof). Hence, each associated coefficient matrix will be equivalent to the stiffness matrix of a beam element, and the conventional numerical assembly technique for the finite element method (FEM) may be used to determine the "overall" coefficient matrix $[\bar{B}]$. Therefore, the eigenvalue equation $[\bar{B}]\{\bar{C}\} = \mathbf{0}$ is easily obtained.

1. INTRODUCTION

Although the analytical solution for the natural frequencies and mode shapes of a uniform beam or plate may be obtained with no difficulty [1], trouble arises when the beam or plate is attached by any kind of "concentrated elements", such as elastically mounted point masses, rigidly attached point masses, translational

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springs, and/or rotational springs [2–22]. Since the natural frequencies of a beam or plate carrying a spring–mass system may deviate considerably from those of the beam or plate itself, a lot of researchers have devoted themselves to study this problem. The solution of this problem has usually been obtained by means of the exact analysis [2–5], or the numerical (approximate) method [6–9].

In theory, most of the approaches presented in the foregoing literature may be extended to solve the eigenvalue problems for a uniform beam or plate carrying "any number of" concentrated elements. In practice, however, they are not easily implemented because of the complexity of the mathematical expressions. For this reason, the total number of "concentrated elements" illustrated in references [1–17] is less than "two".

To circumvent the drawback of the existing approaches, this paper presents a numerical assembly technique to derive the eigenvalue equation $[\bar{B}]\{\bar{C}\} = \mathbf{0}$, and then the conventional methods (e.g., half-interval method and Gauss–Jordan reduction method) were used to solve the eigenvalues and the corresponding eigenvectors. By considering the two (left and right) "sides" of any attaching point for a spring–mass system to the uniform beam and the two (left and right) "ends" of the uniform beam as *nodal points*, the associated integration constants $\bar{C}_i$ $(i = 1 \sim 5n + 4)$ and mode displacement Z_v $(v = 1 \sim n)$ can be viewed as *nodal displacements*, so that the associated coefficient matrix $[B_L]$, $[B_v]$ or $[B_R]$ may now be considered as the element stiffness matrix of a beam element and the conventional assembly technique of the direct stiffness matrix method for the finite element method (FEM) [23] may be used to obtain the "overall" coefficient matrix $[\bar{B}]$. Any trial value of $\bar{\omega}$ that renders the value of the determinant of the overall coefficient matrix equal to zero (i.e., $|\bar{B}| = 0$) represents one of the natural frequencies of the "constrained" beam (i.e., *the uniform beam together with all the attached spring–mass systems*), and the substitution of each corresponding value of $\bar{C}_j$ $(j = 1 \sim 4)$ into the eigenfunction determines the associated mode shapes.

To show the utility of the present approach, the lowest five natural frequencies and mode shapes of a uniform beam carrying one, three and five spring–mass systems, respectively, were calculated. In each case, the four boundary conditions of the uniform beam were studied: clamped–free, simply supported–simply supported, clamped–clamped, and clamped–simply supported. It was found that the agreement between some of the present results and the existing ones was good.

2. FORMULATION OF THE PROBLEM

2.1. EQUATIONS OF MOTION AND THE COMPATIBLE CONDITIONS

Figure 1 shows a cantilever beam carrying n spring–mass systems. The whole cantilever beam with length ℓ is subdivided into $(n + 1)$ segments by the attaching point ⓥ located at $x = x_v$ $(v = 1, 2, \ldots, n)$, where ⓥ denotes the vth "attaching point" and (v) denotes the vth "segment". In addition, the left and right ends of the *beam* are denoted by Ⓛ and Ⓡ, respectively.

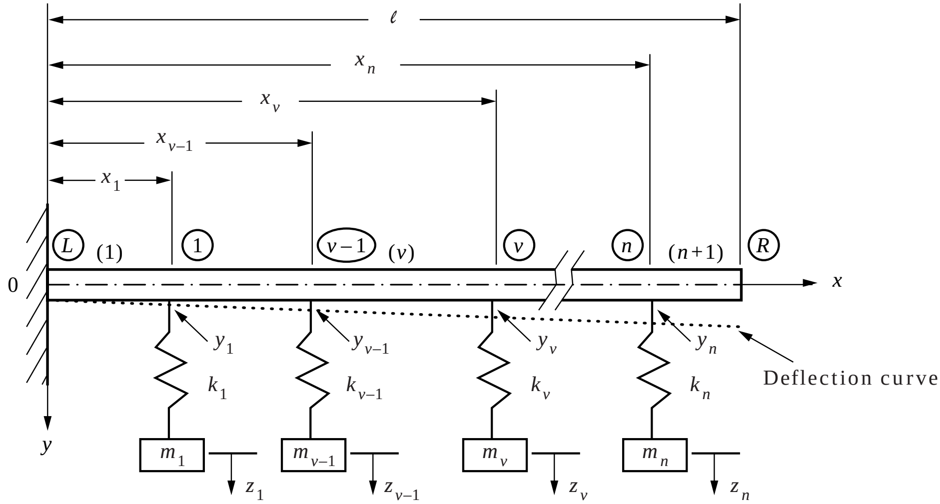

Figure 1. A cantilever beam carrying n spring–mass systems.

If the equations of motion of the "constrained" beam and the vth sprung mass are respectively represented by

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \bar{m}\frac{\partial^2 y(x,t)}{\partial t^2} = 0 \tag{1}$$

$$m_v \ddot{z}_v + k_v(z_v - y_v) = 0 \quad \text{or} \quad m_v \ddot{z}_v = -k_v(z_v - y_v) \tag{2}$$

then the compatibility of the deflection of the "constrained" beam at the vth attaching point requires that

$$y_v^L(x_v, t) = y_v^R(x_v, t), \tag{3}$$

$$y_v'^L(x_v, t) = y_v'^R(x_v, t), \tag{4}$$

$$y_v''^L(x_v, t) = y_v''^R(x_v, t), \tag{5}$$

where E is the Young's modulus, I is the moment of inertia of the cross-sectional area, $\bar{m}$ is the beam mass per unit length, m_v and k_v represent the point mass and spring constant of the vth spring–mass system, $\ddot{z}_v$ and z_v are the acceleration and displacement of the vth sprung mass (relative to its static equilibrium position), y_v, y_v' and y_v'' are the deflection, slope and curvature of the "constrained" beam at the vth attaching point, and the superscripts "L" and "R" represent the left side and right side of the vth attaching point, respectively.

The force equilibrium between the beam and the sprung mass requires that

$$EIy_v'''^L(x_v, t) - EIy_v'''^R(x_v, t) = m_v \ddot{z}_v. \tag{6}$$

The boundary conditions for the cantilever beam are given by

$$y(0, t) = 0, \qquad y'(0, t) = 0, \tag{7a, b}$$

$$y''(\ell, t) = 0, \qquad y'''(\ell, t) = 0, \tag{7c, d}$$

2.2. DERIVATION OF FREQUENCY EQUATION

When the "constrained" beam undergoes free vibration, the instantaneous displacements of the beam and the sprung masses take the form

$$y(x, t) = Y(x)\, \mathrm{e}^{i\bar{\omega}t} \tag{8}$$

$$z_v = Z_v\, \mathrm{e}^{i\bar{\omega}t}, \qquad v = 1, 2, \ldots, n, \tag{9}$$

where $Y(x)$ and Z_v represent the amplitudes of y and z_v, respectively, and $\bar{\omega}$ represents the natural frequency of the "constrained" beam.

The substitution of equation (8) into equation (1) gives

$$Y''''(x) - \beta^4 Y(x) = 0, \tag{10}$$

where

$$\beta^4 = \bar{m}\bar{\omega}^2/EI \quad \text{or} \quad \bar{\omega}^2 = (\beta\ell)^4\left(\frac{EI}{\bar{m}\ell^4}\right), \tag{11}$$

and from equations (2), (8) and (9) one obtains

$$k_v Y_v - (k_v - m_v\bar{\omega}^2)Z_v = 0$$

or

$$Y_v + (-1 + \gamma_v^2)Z_v = 0, \tag{12}$$

where

$$\gamma_v = \bar{\omega}/\omega_v, \qquad \omega_v = \sqrt{k_v/m_v}\,. \tag{13}$$

Similarly, when equations (8) and (9) are introduced into equations (3)–(7), one obtains

$$Y_v^L(x_v) = Y_v^R(x_v), \qquad Y_v'^L(x_v) = Y_v'^R(x_v), \qquad Y_v''^L(x_v) = Y_v''^R(x_v) \tag{(3)$', (4)$', (5)$'$}$$

$$Y_v'''^L(x_v) - Y_v'''^R(x_v) + \frac{m_v^*}{\ell^3}(\beta\ell)^4 Z_v = 0 \tag{(6)$'$}$$

where

$$m_v^* = m_v/m_b, \qquad m_b = \bar{m}\ell \tag{14}$$

and

$$Y(0) = 0, \qquad Y'(0) = 0, \tag{(7a)$'$, (7b)$'$}$$

$$Y''(\ell) = 0, \qquad Y'''(\ell) = 0. \tag{(7c)$'$, (7d)$'$}$$

The solution of equation (10) takes the form

$$Y(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x. \tag{15}$$

For "the vth segment" the last equation may be rewritten as

$$Y_v(\xi) = C_{v1} \sin \beta\ell\xi + C_{v2} \cos \beta\ell\xi + C_{v3} \sinh \beta\ell\xi + C_{v4} \cosh \beta\ell\xi, \tag{16}$$

where

$$\xi = x/\ell. \tag{17}$$

From equations (15)–(17) one has

$$Y_v'(x) = \frac{1}{\ell} Y_v'(\xi) = \beta[C_{v1} \cos \beta\ell\xi - C_{v2} \sin \beta\ell\xi + C_{v3} \cosh \beta\ell\xi + C_{v4} \sinh \beta\ell\xi], \tag{18}$$

$$Y_v''(x) = \frac{1}{\ell^2} Y_v''(\xi) = \beta^2[-C_{v1} \sin \beta\ell\xi - C_{v2} \cos \beta\ell\xi + C_{v3} \sinh \beta\ell\xi + C_{v4} \cosh \beta\ell\xi], \tag{19}$$

$$Y_v'''(x) = \frac{1}{\ell^3} Y'''_v(\xi) = \beta^3[-C_{v1} \cos \beta\ell\xi + C_{v2} \sin \beta\ell\xi + C_{v3} \cosh \beta\ell\xi + C_{v4} \sinh \beta\ell\xi], \tag{20}$$

where $\xi_v \leqslant \xi \leqslant \xi_{v+1}$.

From Figure 1 one sees that the left end of the *beam*, Ⓛ, coincides with the left end of the first *segment* ($v = 1$), hence, from equations (7a)′, (7b)′ and (16)–(18) one obtains

$$Y(0) = C_{12} + C_{14} = 0, \tag{21a}$$

$$Y'(0) = \beta[C_{11} + C_{13}] = 0 \quad \text{or} \quad C_{11} + C_{13} = 0. \tag{21b}$$

To write equations (21a) and (21b) in matrix form gives

$$[B_L]\{C_L\} = \mathbf{0}, \tag{22a}$$

where

$$[B_L] = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} & \\ & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \end{matrix} \end{matrix}, \tag{22b}$$

$$\{C_L\} = \{C_{11} \quad C_{12} \quad C_{13} \quad C_{14}\} = \{\bar{C}_1 \quad \bar{C}_2 \quad \bar{C}_3 \quad \bar{C}_4\}, \tag{22c}$$

where the symbols [] and {} represent the rectangular matrix and the column vector, respectively, and

$$\bar{C}_1 = C_{11}, \qquad \bar{C}_2 = C_{12}, \qquad \bar{C}_3 = C_{13}, \qquad \bar{C}_4 = C_{14}. \tag{22d}$$

In equation (22b) and the subsequent equations, the numbers shown on the top side and right side of the matrix represent the identification numbers of degrees of freedom (dof) for the associated constants $\bar{C}_i$ ($i = 1, 2, \ldots$).

For the vth attaching point, from equations (3)′–(6)′ and equations (16)–(20), one obtains

$$C_{v1}\mathbf{s}\theta_v + C_{v2}\mathbf{c}\theta_v + C_{v3}\mathbf{sh}\theta_v + C_{v4}\mathbf{ch}\theta_v - C_{v+1,1}\mathbf{s}\theta_v - C_{v+1,2}\mathbf{c}\theta_v - C_{v+1,3}\mathbf{sh}\theta_v - C_{v+1,4}\mathbf{ch}\theta_v = 0, \tag{23}$$

$$C_{v1}\mathbf{c}\theta_v - C_{v2}\mathbf{s}\theta_v + C_{v3}\mathbf{ch}\theta_v + C_{v4}\mathbf{sh}\theta_v - C_{v+1,1}\mathbf{c}\theta_v + C_{v+1,2}\mathbf{s}\theta_v - C_{v+1,3}\mathbf{ch}\theta_v - C_{v+1,4}\mathbf{sh}\theta_v = 0, \tag{24}$$

$$-C_{v1}\mathbf{s}\theta_v - C_{v2}\mathbf{c}\theta_v + C_{v3}\mathbf{sh}\theta_v + C_{v4}\mathbf{ch}\theta_v + C_{v+1,1}\mathbf{s}\theta_v + C_{v+1,2}\mathbf{c}\theta_v - C_{v+1,3}\mathbf{sh}\theta_v - C_{v+1,4}\mathbf{ch}\theta_v = 0, \tag{25}$$

$$[-C_{v1}\mathbf{c}\theta_v + C_{v2}\mathbf{s}\theta_v + C_{v3}\mathbf{ch}\theta_v + C_{v4}\mathbf{sh}\theta_v + C_{v+1,1}\mathbf{c}\theta_v - C_{v+1,2}\mathbf{s}\theta_v - C_{v+1,3}\mathbf{ch}\theta_v - C_{v+1,4}\mathbf{sh}\theta_v] + m_v^*(\beta\ell)Z_v = 0. \tag{26}$$

Substituting equation (16) into equation (12) yields one additional relationship

$$C_{v1}\mathbf{s}\theta_v + C_{v2}\mathbf{c}\theta_v + C_{v3}\mathbf{sh}\theta_v + C_{v4}\mathbf{ch}\theta_v + (-1 + \gamma_v^2)Z_v = 0, \tag{27}$$

where

$$\mathbf{s}\theta_v = \sin\beta\ell\xi_v, \qquad \mathbf{c}\theta_v = \cos\beta\ell\xi_v, \qquad \mathbf{sh}\theta_v = \sinh\beta\ell\xi_v, \qquad \mathbf{ch}\theta_v = \cosh\beta\ell\xi_v \tag{28a}$$

and

$$\theta_v = \beta\ell\xi_v. \tag{28b}$$

It is noted that in equations (3)′–(6)′, the "left side" of the vth attaching point located at $x = x_v$ (or $\xi = \xi_v$) belongs to the segment v and the "right side" belongs to the segment $v + 1$. Thus the associated coefficients are represented by C_{vj} and $C_{v+1,j}$ ($j = 1 \sim 4$), respectively. Equations (23)–(27) in matrix form gives

$$[B_v]\{C_v\} = \mathbf{0}, \tag{29}$$

$$
\begin{array}{c}
\begin{array}{ccccccccc}
4v-3 & 4v-2 & 4v-1 & 4v & 4v+1 & 4v+2 & 4v+3 & 4v+4 & 4v+5
\end{array}
\\
[B_v] = \left[\begin{array}{ccccccccc}
\mathbf{s}\theta_v & \mathbf{c}\theta_v & \mathbf{sh}\theta_v & \mathbf{ch}\theta_v & -\mathbf{s}\theta_v & -\mathbf{c}\theta_v & -\mathbf{sh}\theta_v & -\mathbf{ch}\theta_v & 0 \\
\mathbf{c}\theta_v & -\mathbf{s}\theta_v & \mathbf{ch}\theta_v & \mathbf{sh}\theta_v & -\mathbf{c}\theta_v & \mathbf{s}\theta_v & -\mathbf{ch}\theta_v & -\mathbf{sh}\theta_v & 0 \\
-\mathbf{s}\theta_v & -\mathbf{c}\theta_v & \mathbf{sh}\theta_v & \mathbf{ch}\theta_v & \mathbf{s}\theta_v & \mathbf{c}\theta_v & -\mathbf{sh}\theta_v & -\mathbf{ch}\theta_v & 0 \\
-\mathbf{c}\theta_v & \mathbf{s}\theta_v & \mathbf{ch}\theta_v & \mathbf{sh}\theta_v & \mathbf{c}\theta_v & -\mathbf{s}\theta_v & -\mathbf{ch}\theta_v & -\mathbf{sh}\theta_v & \mathbf{m}_v^*(\beta\ell) \\
\mathbf{s}\theta_v & \mathbf{c}\theta_v & \mathbf{sh}\theta_v & \mathbf{ch}\theta_v & 0 & 0 & 0 & 0 & -1+\gamma_v^2
\end{array}\right]
\begin{array}{l}
5v-2 \\ 5v-1 \\ 5v \\ 5v+1 \\ 5v+2
\end{array}
\end{array}
\qquad (30)
$$

$$\{C_v\} = \{C_{v1} \quad C_{v2} \quad C_{v3} \quad C_{v4} \quad C_{v+1,1} \quad C_{v+1,2} \quad C_{v+1,3} \quad C_{v+1,4} \quad Z_v\}$$
$$= \{\bar{C}_{4v-3} \quad \bar{C}_{4v-2} \quad \bar{C}_{4v-1} \quad \bar{C}_{4v} \quad \bar{C}_{4v+1} \quad \bar{C}_{4v+2} \quad \bar{C}_{4v+3} \quad \bar{C}_{4v+4} \quad \bar{C}_{4v+5}\}, \tag{31}$$

where

$$\bar{C}_{4v-3} = C_{v1}, \qquad \bar{C}_{4v-2} = C_{v2}, \ldots, \qquad \bar{C}_{4v+4} = C_{v+1,4}, \qquad \bar{C}_{4v+5} = Z_v. \tag{32}$$

Since the right end of the beam, Ⓡ, coincides with the right end of the $(n+1)$th segment ($v = n+1$), as may be seen from Figure 1, hence from equations (7c)′, (7d)′ and (19), (20) one obtains

$$-C_{n+1,1}\sin\beta\ell - C_{n+1,2}\cos\beta\ell + C_{n+1,3}\sinh\beta\ell + C_{n+1,4}\cosh\beta\ell = 0, \tag{33a}$$

$$-C_{n+1,1}\cos\beta\ell - C_{n+1,2}\sin\beta\ell + C_{n+1,3}\cosh\beta\ell + C_{n+1,4}\sinh\beta\ell = 0, \tag{33b}$$

or

$$[B_R]\{C_R\} = \mathbf{0}, \tag{34}$$

where

$$[B_R] = \begin{bmatrix} 4n+1 & 4n+2 & 4n+3 & 4n+4 \\ -\sin\beta\ell & -\cos\beta\ell & \sinh\beta\ell & \cosh\beta\ell \\ -\cos\beta\ell & \sin\beta\ell & \cosh\beta\ell & \sinh\beta\ell \end{bmatrix} \begin{matrix} \\ p-1, \\ p \end{matrix} \tag{35}$$

$$\{C_R\} = \{C_{n+1,1} \quad C_{n+1,2} \quad C_{n+1,3} \quad C_{n+1,4}\}$$
$$= \{\bar{C}_{4n+1} \quad \bar{C}_{4n+2} \quad \bar{C}_{4n+3} \quad \bar{C}_{4n+4}\}, \tag{36}$$

$$\bar{C}_{4n+1} = C_{n+1,1}, \qquad \bar{C}_{4n+2} = C_{n+1,2}, \qquad \bar{C}_{4n+3} = C_{n+1,3}, \qquad \bar{C}_{4n+4} = C_{n+1,4}, \tag{37}$$

$$p = 5n + 4. \tag{38}$$

In the last equations, p represents the total number of equations. From the above derivations one sees that from each attaching point for a spring–mass system one may obtain five equations (including three compatibility equations, one force-equilibrium equation and one governing equation for the sprung mass), and from each boundary (Ⓛ or Ⓡ) one may obtain two equations. Hence, for a beam carrying n spring–mass systems, the total number of equations that one may obtain for the integration constants C_{vi} and mode displacements Z_v ($v = 1 \sim n$, $i = 1 \sim 4$) is equal to $5n + 4$, i.e., $p = 5n + 4$ as shown by equation (38). Of course, the total number of unknowns (C_{vi} and Z_v) is also equal to $5n + 4$. From equation (16) one sees that the solution $Y_v(\xi)$ for each beam segment contains four unknown integration constants C_{vi} ($i = 1 \sim 4$), and from equations (12) and (6)′ one sees that the governing equation or the force-equilibrium equation for each sprung mass contains one additional unknown Z_v. Hence if a beam carries n sprung masses, the total number of the beam segment is $n + 1$ and thus the total number of unknowns (C_{vi} and Z_v) is equal to $4(n+1) + n = 5n + 4 = p$.

If all the unknowns C_{vi} and Z_i ($v = 1 \sim n$, $i = 1 \sim 4$) are replaced by a column vector $\{\bar{C}\}$ with coefficients $\bar{C}_j$ ($j = 1, 2, \ldots, p$) defined by equations (22d), (32) and (37), then the matrices $[B_L]$, $[B_v]$ and $[B_R]$ are similar to the element property matrices (for the finite element method) with corresponding identification numbers of degrees of freedom (dof) shown on the top side and right side of the matrices defined by equations (22b), (30) and (35). Based on the assembly technique for the direct stiffness matrix method, it is easy to arrive at the following coefficient equation for the entire vibrating system

$$[\bar{B}]\{\bar{C}\} = \mathbf{0}. \tag{39}$$

The non-trivial solution of the problem requires that

$$|\bar{B}| = 0, \tag{40}$$

which is the frequency equation, and the half-interval technique [24] may be used to solve the eigenvalues $\bar{\omega}_i$ ($i = 1, 2, \ldots$).

To substitute each value of $\bar{\omega}_i$ into equation (39) one may determine the values of unknowns $\bar{C}_i$ ($i = 1 \sim p$). Among which, $\bar{C}_1 = C_1$, $\bar{C}_2 = C_2$, $\bar{C}_3 = C_3$, $\bar{C}_4 = C_4$. Hence the substitution of C_j ($j = 1 \sim 4$) into equation (15) will define the corresponding mode shape $Y^{(i)}(\xi)$. For a cantilever beam carrying one ($n = 1$) and two ($n = 2$) spring–mass systems, the corresponding matrices $[\bar{B}]_{(1)}$ and $[\bar{B}]_{(2)}$ are shown in the Appendix [see equations (A1) and (A2)]. From the lengthy expressions one sees that the conventional explicit formulations are not suitable for a beam carrying more than three ($n = 3$) spring–mass systems. However, the numerical assembly technique presented in this paper may easily solve this problem.

3. DETERMINATION OF $[B_L]$ and $[B_R]$ FOR VARIOUS SUPPORTING CONDITIONS

From the last section one finds that the forms of $[B_v]$ for each attaching point of the spring–mass system have nothing to do with the boundary conditions of the carrying beam. Hence, for a "constrained" beam with various supporting conditions, the only thing one should do is to modify the values of the two boundary matrices $[B_L]$ and $[B_R]$ given by equations (22b) and (35), respectively, according to the associated boundary conditions. Then the same assembly procedures presented in the last section may be followed. This is one of the predominant advantages of the presented approach. The boundary matrices $[B_L]$ and $[B_R]$ for various boundary conditions are listed below:

(1) Hinged–hinged beam

$$[B_L] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \end{matrix}, \tag{41}$$

$$[B_R] = \begin{array}{c} \begin{matrix} 4n+1 & 4n+2 & 4n+3 & 4n+4 \end{matrix} \\ \begin{bmatrix} \sin\beta\ell & \cos\beta\ell & \sinh\beta\ell & \cosh\beta\ell \\ -\sin\beta\ell & -\cos\beta\ell & \sinh\beta\ell & \cosh\beta\ell \end{bmatrix} \end{array} \begin{matrix} p-1 \\ p \end{matrix}. \tag{42}$$

(2) Clamped–clamped beam

$$[B_L] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \end{matrix}, \tag{43}$$

$$[B_R] = \begin{array}{c} \begin{matrix} 4n+1 & 4n+2 & 4n+3 & 4n+4 \end{matrix} \\ \begin{bmatrix} \sin\beta\ell & \cos\beta\ell & \sinh\beta\ell & \cosh\beta\ell \\ \cos\beta\ell & -\sin\beta\ell & \cosh\beta\ell & \sinh\beta\ell \end{bmatrix} \end{array} \begin{matrix} p-1 \\ p \end{matrix}. \tag{44}$$

(3) Clamped–hinged beam

$$[B_L] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \end{matrix}, \tag{45}$$

$$[B_R] = \begin{array}{c} \begin{matrix} 4n+1 & 4n+2 & 4n+3 & 4n+4 \end{matrix} \\ \begin{bmatrix} \sin\beta\ell & \cos\beta\ell & \sinh\beta\ell & \cosh\beta\ell \\ -\sin\beta\ell & -\cos\beta\ell & \sinh\beta\ell & \cosh\beta\ell \end{bmatrix} \end{array} \begin{matrix} p-1 \\ p \end{matrix}, \tag{46}$$

4. NUMERICAL RESULTS AND DISCUSSION

The dimensions and physical properties for the uniform beam studied here are: $\ell = 1{\cdot}0$ m, $d = 0{\cdot}05$ m, $E = 2{\cdot}069 \times 10^{11}$ N/m^2, $\rho = 7{\cdot}8367 \times 10^3$ kg/m^3, $\bar{m} = \rho A = 15{\cdot}3875$ kg/m, $I = \pi d^4/64 = 3{\cdot}06796 \times 10^{-7}$ m^4, $m_b = \bar{m}\ell = 15{\cdot}3875$ kg, $k_b = EI/\ell^3 = 6{\cdot}34761 \times 10^4$ N/m. It is worth mentioning that m_b represents the total mass of the beam, and k_b represents one-third (1/3) of the spring constant of a clamped–free beam at the free end. Since m_b and k_b are the important mass parameter and stiffness parameter of a uniform beam, respectively, they are used as the bases of dimensionless parameters m_i^* $(=m_i/m_b)$ and k_i^* $(=k_i/k_b)$, $i = 1, 2, \ldots$ in the following discussion.

4.1. RELIABILITY OF THE THEORY AND THE COMPUTER PROGRAMS

In the existing literature, only the case of a uniform beam or plate carrying "one" spring–mass system can be found [2–8]. For example, reference [3] determined the natural frequencies of a cantilever beam carrying "one" elastically mounted point mass at $\xi_1 = x_1/\ell = 0{\cdot}75$, and the dimensionless spring constant and point mass were: $k_1^* = k_1/k_b = 3{\cdot}0$ and $m_1^* = m_1/m_b = 0{\cdot}2$. A similar problem was also studied in reference [2], but the sprung mass was located at the free end (i.e., $\xi_1 = 1{\cdot}0$) and the dimensionless magnitudes of spring constant and point mass were: $k_1^* = 100{\cdot}0$ and $m_1^* = 0{\cdot}5$. The lowest five natural frequencies of the "constrained" cantilever beam, $\bar{\omega}_i$ ($i = 1 \sim 5$) (rad/s), are shown in Table 1. From Table 1 one sees that the values of $\bar{\omega}_i$ ($i = 1 \sim 5$) obtained from the present method

TABLE 1

The lowest five natural frequencies $\bar{\omega}_i$ ($i = 1 \sim 5$) for a cantilever beam carrying "one" spring–mass system

Location $\xi_1 = x_1/\ell$	$k_1^* = \frac{k_1}{k_b}$	$m_1^* = \frac{m_1}{m_b}$	Methods	Natural frequencies (rad/s) $\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
0·75	3·0	0·2	Present	174·2030	322·1513	1415·5524	3964·7796	7766·4614
			Reference [3]	174·2097	322·1653	1415·5823	3964·9742	7766·7348
1·0	100	0·5	Present	128·6160	971·9372	2131·4107	4210·0351	7879·2394
			Reference [2]	128·6211	–	–	–	–

Note: $\ell = 1{\cdot}0$ m; $k_b = EI/\ell^3 = 6{\cdot}34761 \times 10^4$ N/m; $m_b = \bar{m}\ell = 15{\cdot}3875$ kg.

TABLE 2

The lowest five natural frequencies $\bar{\omega}_i$ ($i = 1 \sim 5$) for a uniform beam carrying "one" spring–mass system

Boundary conditions	Location $\xi_1 = x_1/\ell$	$k_1^* = \frac{k_1}{k_b}$	$m_1^* = \frac{m_1}{m_b}$	Natural frequencies (rad/s) $\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
CF				174·2030	322·1513	1415·5524	3964·7796	7766·4614
SS	0·75	3·0	0·2	243·8579	645·2030	2540·5306	5706·1886	10142·4012
CC				247·9185	1440·2751	3964·3645	7766·8230	12836·6308
CS				245·9788	1000·1317	3212·8284	6696·1421	11449·8613

Note: $\ell = 1{\cdot}0$ m; $k_b = EI/\ell^3 = 6{\cdot}34761 \times 10^4$ N/m; $m_b = \bar{m}\ell = 15{\cdot}3875$ kg.

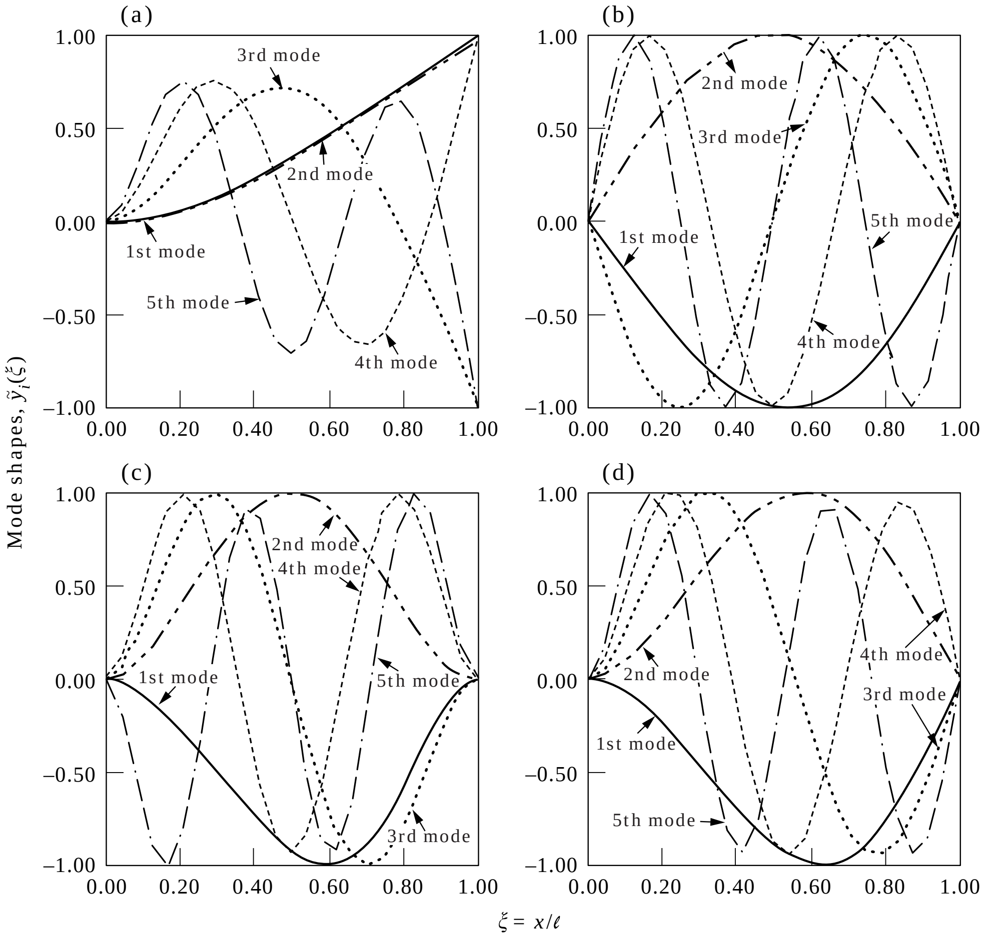

Figure 2. The lowest five mode shapes $\tilde{y}_i(\xi)$ ($i = 1 \sim 5$) for a uniform beam carrying "one" spring–mass system with spring constant $k_1 = 3K_b = 1{\cdot}90428 \times 10^5$ N/m and point mass $m_1 = 0{\cdot}2m_b = 3{\cdot}0775$ kg located at $\xi_1 = x_1/\ell = 0{\cdot}75$ for the support conditions. (a) CF, (b) SS, (c) CC and (d) CS.

TABLE 3

The lowest five natural frequencies $\bar{\omega}_i$ ($i = 1 \sim 5$) for the uniform beam carrying "three" spring–mass systems ($k_1 = 3k_b$, $m_1 = 0{\cdot}2m_b$; $k_2 = 4{\cdot}5k_b$, $m_2 = 0{\cdot}5m_b$; $k_3 = 6k_b$, $m_3 = 1{\cdot}0m_b$)

Boundary conditions	Locations of the three spring-mass systems $\xi_i = x_i/\ell$			Natural frequencies (rad/s)				
	ξ_1	ξ_2	ξ_3	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
CF				102·7994	188·7347	248·6439	349·1161	1427·9521
SS	0·1	0·4	0·8	152·7341	185·0950	247·8314	677·5961	2548·6577
CC				156·6703	190·6994	248·6622	1454·2932	3968·4732
CS				154·6730	189·8148	248·6554	1017·8438	3211·9416

Note: $\ell = 1{\cdot}0$ m; $k_b = EI/\ell^3 = 6{\cdot}34761 \times 10^4$ N/m; $m_b = \bar{m}\ell = 15{\cdot}3875$ kg.

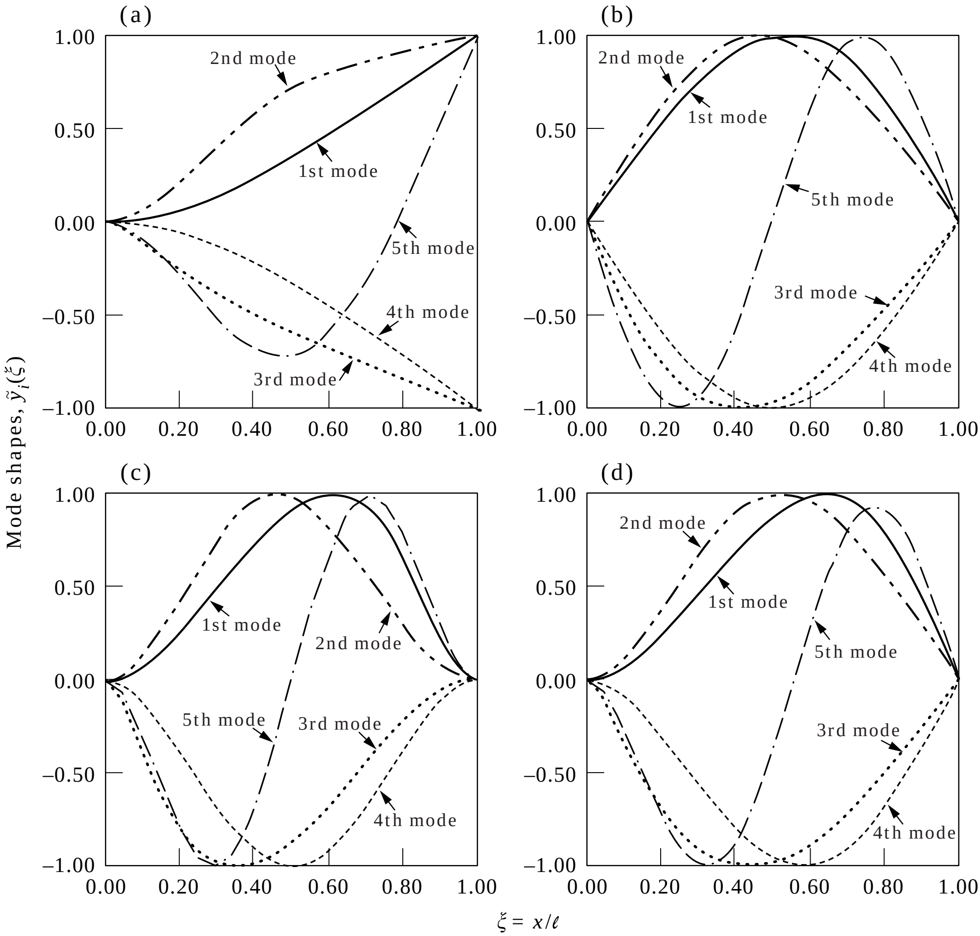

Figure 3. The lowest five mode shapes $\tilde{y}_i$ $(\xi)(i = 1 \sim 5)$ for a uniform beam carrying "three" spring–mass systems with spring constants $k_1 = 3k_b$, $k_2 = 4{\cdot}5k_b$, $k_3 = 6k_b$ and point mass $m_1 = 0{\cdot}2m_b$, $m_2 = 0{\cdot}5m_b$, $m_3 = 1{\cdot}0m_b$ located at $\xi_1 = x_1/\ell = 0{\cdot}1$, $\xi_2 = x_2/\ell = 0{\cdot}4$, $\xi_3 = x_3/\ell = 0{\cdot}8$ for the support conditions: (a) CF, (b) SS, (c) CC and (d) CS, respectively.

TABLE 4

The locations and magnitudes of the "five" spring–mass systems on a uniform beam

Locations $\xi_i = x_i/\ell$					Magnitudes of spring constants $k_i^* = k_i/k_b$					Magnitudes of point masses $m_i^* = m_i/m_b$				
ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	k_1^*	k_2^*	k_3^*	k_4^*	k_5^*	m_1^*	m_2^*	m_3^*	m_4^*	m_5^*
0·1	0·2	0·4	0·6	0·8	3	3·5	4·5	5	6	0·2	0·3	0·5	0·65	1·0

Note: $\ell = 1{\cdot}0$ m; $k_b = EI/\ell^3 = 6{\cdot}34761 \times 10^4$ N/m; $m_b = \bar{m}\ell = 15{\cdot}3875$ kg.

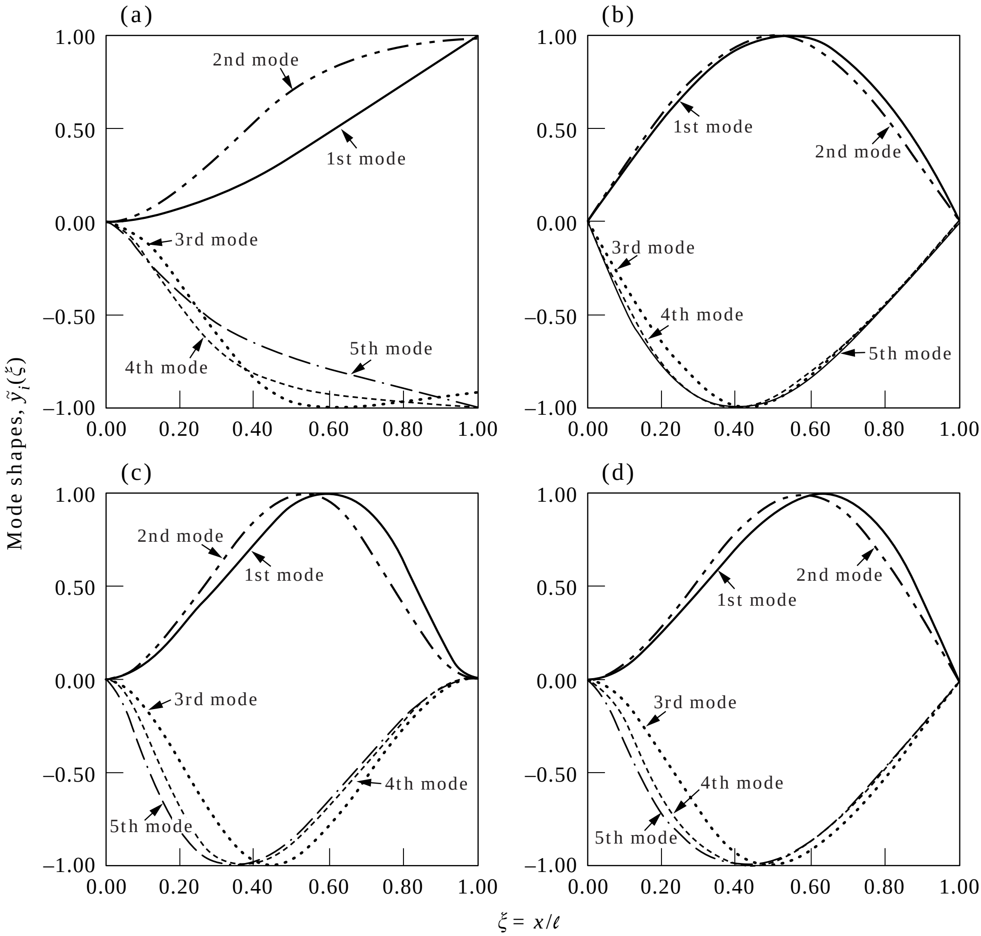

Figure 4. The lowest five mode shapes $\tilde{y}_i(\xi)$ $(i = 1 \sim 5)$ for a uniform beam carrying "five" spring–mass systems with locations and magnitudes shown in Table 4 for the support conditions: (a) CF, (b) SS, (c) CC and (d) CS, respectively.

are very close to those from reference [3] or [2]. Hence, the reliability of the theory presented and the computer programs developed in this paper should be acceptable. It is noted that the eigenvalues presented in references [3] and [2] are the frequency coefficients $\overline{\beta}_i\ell$, and those shown in Table 1 are the actual natural frequencies $\bar{\omega}_i$, the relationship between them given by $\bar{\omega}_i = (\overline{\beta}_i\ell)^2 \sqrt{EI/\bar{m}\ell^4}$ $(i = 1, 2, \ldots)$.

4.2. A UNIFORM BEAM CARRYING ANY NUMBER OF SPRING–MASS SYSTEMS

In this section, the presented method is used to calculate the lowest five natural frequencies and the corresponding mode shapes of a uniform beam carrying "any number of" sprung masses, to demonstrate the availability of the method for a general constrained beam. Four boundary (/and supported) conditions of the utility constrained beam are studied here. For convenience, a two-letter acronym

TABLE 5

The lowest five natural frequencies $\bar{\omega}_i$ ($i = 1 \sim 5$) for a uniform beam carrying "five" spring–mass systems

Boundary conditions	Natural frequencies (rad/s)				
	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
CF	97·4880	171·6770	190·1782	218·7927	248·6537
SS	150·9571	169·4729	187·9147	217·1279	247·9868
CC	156·6318	175·9684	190·8985	218·8880	248·6641
CS	154·2684	173·8962	190·4056	218·8299	248·6588

Note: $\ell = 1{\cdot}0$ m; $k_b = EI/\ell^3 = 6{\cdot}34761 \times 10^4$ N/m; $m_b = \bar{m}\ell = 15{\cdot}3875$ kg.

is used to designate the type of support, starting from the left end to the right end. Hence, if the clamped, free and simply supported ends are denoted by C, F and S, respectively, the boundary conditions of Tables 2, 3 and 5 may be represented by CF, SS, CC and CS, respectively.

For the case of a uniform beam carrying "one" sprung mass located at $\xi_1 = 0{\cdot}75$, if the spring constant and point mass are $k_1 = 3k_b$, and $m_1 = 0{\cdot}2m_b$, then the lowest five natural frequencies $\bar{\omega}_i$ ($i = 1 \sim 5$) and the corresponding mode shapes $\tilde{y}_i(\xi)$ of the uniform "constrained" beam are shown in Table 2 and Figure 2, respectively.

For the case of a uniform beam carrying "three" sprung masses located at $\xi_1 = 0{\cdot}1$, $\xi_2 = 0{\cdot}4$ and $\xi_3 = 0{\cdot}8$, where the respective spring constants of the three spring–mass systems are $k_1 = 3k_b$, $k_2 = 4{\cdot}5k_b$ and $k_3 = 6k_b$, and the respective magnitudes of the three sprung masses are $m_1 = 0{\cdot}2m_b$, $m_2 = 0{\cdot}5m_b$ and $m_3 = 1{\cdot}0m_b$, the lowest five natural frequencies $\bar{\omega}_i$ ($i = 1 \sim 5$) and the corresponding mode shapes $\tilde{y}_i(\xi)$ of the uniform "constrained" beam are shown in Table 3 and Figure 3, respectively.

For the case of a uniform beam carrying "five" spring–mass systems with the locations and magnitudes of the five springs and five sprung masses summarized in Table 4, the lowest five natural frequencies $\bar{\omega}_i$ ($i = 1 \sim 5$) and the corresponding mode shapes $\tilde{y}_i(\xi)$ are shown in Table 5 and Figure 4 respectively.

It is noted that the mode shapes $\tilde{y}_i(\xi)$ shown in Figures 2–4 are normalized such that the maximum value of each mode is equal to unity.

5. CONCLUSION

For a uniform beam carrying more than "two" spring–mass systems, the determination of exact solutions for the natural frequencies and mode shapes of the "constrained" beam is usually not easy. However, by means of the technique presented in this paper, one may obtain the exact natural frequencies and mode shapes of a uniform beam carrying "any number of" spring–mass systems with no difficulty.

ACKNOWLEDGMENT

The authors wish to thank Professor Metin Gürgöze of the Faculty of Mechanical Engineering, Istanbul Technical University, for his valuable comments.

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APPENDIX

For a cantilever beam carrying one ($n = 1$) and two ($n = 2$) spring–mass systems, the corresponding coefficient matrices $[\bar{B}]_{(1)}$ and $[\bar{B}]_{(2)}$ are

$$[B_v]_{(1)} =$$

$$\begin{array}{c}
\begin{array}{ccccccccc}
\bar{C}_1 & \bar{C}_2 & \bar{C}_3 & \bar{C}_4 & \bar{C}_5 & \bar{C}_6 & \bar{C}_7 & \bar{C}_8 & \bar{C}_9
\end{array}\\
\left[\begin{array}{ccccccccc}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
\mathbf{s}\theta_1 & \mathbf{c}\theta_1 & \mathbf{sh}\theta_1 & \mathbf{ch}\theta_1 & -\mathbf{s}\theta_1 & -\mathbf{c}\theta_1 & -\mathbf{sh}\theta_1 & -\mathbf{ch}\theta_1 & 0\\
\mathbf{c}\theta_1 & -\mathbf{s}\theta_1 & \mathbf{ch}\theta_1 & \mathbf{sh}\theta_1 & -\mathbf{c}\theta_1 & \mathbf{s}\theta_1 & -\mathbf{ch}\theta_1 & -\mathbf{sh}\theta_1 & 0\\
-\mathbf{s}\theta_1 & -\mathbf{c}\theta_1 & \mathbf{sh}\theta_1 & \mathbf{ch}\theta_1 & \mathbf{s}\theta_1 & \mathbf{c}\theta_1 & -\mathbf{sh}\theta_1 & -\mathbf{ch}\theta_1 & 0\\
-\mathbf{c}\theta_1 & \mathbf{s}\theta_1 & \mathbf{ch}\theta_1 & \mathbf{sh}\theta_1 & \mathbf{c}\theta_1 & -\mathbf{s}\theta_1 & -\mathbf{ch}\theta_1 & -\mathbf{sh}\theta_1 & m_1^*(\beta\ell)\\
\mathbf{s}\theta_1 & \mathbf{c}\theta_1 & \mathbf{sh}\theta_1 & \mathbf{ch}\theta_1 & 0 & 0 & 0 & 0 & -1+\gamma_1^2\\
0 & 0 & 0 & 0 & -\mathbf{s}\beta\ell & -\mathbf{c}\beta\ell & \mathbf{sh}\beta\ell & \mathbf{ch}\beta\ell & 0\\
0 & 0 & 0 & 0 & -\mathbf{c}\beta\ell & \mathbf{s}\beta\ell & \mathbf{ch}\beta\ell & \mathbf{sh}\beta\ell & 0
\end{array}\right]
\begin{array}{c}
1\\2\\3\\4\\5\\6\\7\\8\\9
\end{array}
\end{array}$$

(A1)

and

$$
[\bar{B}]_{(2)} = \begin{array}{cccccccccccccc|c}
\bar{C}_1 & \bar{C}_2 & \bar{C}_3 & \bar{C}_4 & \bar{C}_5 & \bar{C}_6 & \bar{C}_7 & \bar{C}_8 & \bar{C}_9 & \bar{C}_{10} & \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{14} & \\
\hline
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\mathbf{s}\theta_1 & \mathbf{c}\theta_1 & \mathbf{sh}\theta_1 & \mathbf{ch}\theta_1 & -\mathbf{s}\theta_1 & -\mathbf{c}\theta_1 & -\mathbf{sh}\theta_1 & -\mathbf{ch}\theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
\mathbf{c}\theta_1 & -\mathbf{s}\theta_1 & \mathbf{ch}\theta_1 & \mathbf{sh}\theta_1 & -\mathbf{c}\theta_1 & \mathbf{s}\theta_1 & -\mathbf{ch}\theta_1 & -\mathbf{sh}\theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
-\mathbf{s}\theta_1 & -\mathbf{c}\theta_1 & \mathbf{sh}\theta_1 & \mathbf{ch}\theta_1 & \mathbf{s}\theta_1 & \mathbf{c}\theta_1 & -\mathbf{sh}\theta_1 & -\mathbf{ch}\theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
-\mathbf{c}\theta_1 & \mathbf{s}\theta_1 & \mathbf{ch}\theta_1 & \mathbf{sh}\theta_1 & \mathbf{c}\theta_1 & -\mathbf{s}\theta_1 & -\mathbf{ch}\theta_1 & -\mathbf{sh}\theta_1 & 0 & 0 & 0 & 0 & m_1^*(\beta\ell) & 0 & 6 \\
\mathbf{s}\theta_1 & \mathbf{c}\theta_1 & \mathbf{sh}\theta_1 & \mathbf{ch}\theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+\gamma_1^2 & 0 & 7 \\
0 & 0 & 0 & 0 & \mathbf{s}\theta_2 & \mathbf{c}\theta_2 & \mathbf{sh}\theta_2 & \mathbf{ch}\theta_2 & -\mathbf{s}\theta_2 & -\mathbf{c}\theta_2 & -\mathbf{sh}\theta_2 & -\mathbf{ch}\theta_2 & 0 & 0 & 8 \\
0 & 0 & 0 & 0 & \mathbf{c}\theta_2 & -\mathbf{s}\theta_2 & \mathbf{ch}\theta_2 & \mathbf{sh}\theta_2 & -\mathbf{c}\theta_2 & \mathbf{s}\theta_2 & -\mathbf{ch}\theta_2 & -\mathbf{sh}\theta_2 & 0 & 0 & 9 \\
0 & 0 & 0 & 0 & -\mathbf{s}\theta_2 & -\mathbf{c}\theta_2 & \mathbf{sh}\theta_2 & \mathbf{ch}\theta_2 & \mathbf{s}\theta_2 & \mathbf{c}\theta_2 & -\mathbf{sh}\theta_2 & -\mathbf{ch}\theta_2 & 0 & 0 & 10 \\
0 & 0 & 0 & 0 & -\mathbf{c}\theta_2 & \mathbf{s}\theta_2 & \mathbf{ch}\theta_2 & \mathbf{sh}\theta_2 & \mathbf{c}\theta_2 & -\mathbf{s}\theta_2 & -\mathbf{ch}\theta_2 & -\mathbf{sh}\theta_2 & 0 & m_2^*(\beta\ell) & 11 \\
0 & 0 & 0 & 0 & \mathbf{s}\theta_2 & \mathbf{c}\theta_2 & \mathbf{sh}\theta_2 & \mathbf{ch}\theta_2 & 0 & 0 & 0 & 0 & 0 & -1+\gamma_2^2 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{s}\beta\ell & -\mathbf{c}\beta\ell & \mathbf{sh}\beta\ell & \mathbf{ch}\beta\ell & 0 & 0 & 13 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{c}\beta\ell & \mathbf{s}\beta\ell & \mathbf{ch}\beta\ell & \mathbf{sh}\beta\ell & 0 & 0 & 14
\end{array}
$$

Where $\mathbf{s}\theta_v = \sin\beta\ell\xi_v$, $\mathbf{c}\theta_v = \cos\beta\ell\xi_v$, $\mathbf{sh}\theta_v = \sinh\beta\ell\xi_v$, $\mathbf{ch}\theta_v = \cosh\beta\ell\xi_v$, $\theta_v = \beta\ell\xi_v$, $m_v^* = m_v/m_b$, $\gamma_v^2 = \bar{\omega}^2/\omega_v^2$ $(v = 1 \sim 2)$ and $\mathbf{s}\beta\ell = \sin\beta\ell$, $\mathbf{c}\beta\ell = \cos\beta\ell$, $\mathbf{sh}\beta\ell = \sinh\beta\ell$, $\mathbf{ch}\beta\ell = \cosh\beta\ell$.